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Dynamic scaling: Distinguishing self-organized from generically critical systems

Maya Paczuski*
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973
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The dynamic scaling approach separates nonequilibrium critical phenomena into two distinct categories: (a) those that are "generically" critical due to symmetry and (b) those that are self-organized critical. This phenomenological approach is demonstrated in the context of interface growth and depinning, where the surface width obeys the scaling form $W(L,s_0,s_0+s)=(s/L^d)^\beta F(s_0/L^D,s/L^D)$. The quantity L is the linear system size, s_0 is the total motion of the interface, and s is the amount of growth separating two configurations. In case (b) the function F has a nontrivial dependence on s_0/L^D reflecting a diverging correlation length, while in case (a) it does not.

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The scaling behavior exhibited by systems that are out of equilibrium brings much attention as a result of its ubiquity in nature and its theoretical challenges. Familiar examples include diffusion limited aggregation (DLA) [1], ballistic deposition [2], sandpiles [3], depinning of interfaces in quenched random media [4], and fluid invasion in a porous medium [5]. All of these dynamical processes generate scale invariant spatial structures. Although many of these systems, such as DLA, are poorly understood, others have yielded to theoretical insight. Ballistic deposition is a prototypical example of kinetic roughening which is known to be described by a Langevin equation proposed by Kardar, Parisi, and Zhang [6]. In ballistic deposition, particles rain down at random onto a surface following straight trajectories, and stick upon the first encounter, either vertically or sideways, with the surface. Similar Langevin equations have been proposed to describe several other systems, including sandpiles [7]. The term "generic scale invariance" was coined for nonequilibrium phenomena that are described by Langevin equations, which are scale invariant from symmetry considerations or conservation laws [8].

On the other hand, Bak, Tang, and Wiesenfeld [3] have proposed that large dynamical systems tend to self-organize, after a long transient, into a critical state (SOC). SOC describes far from equilibrium phenomena that are history dependent. These systems reach their dynamical attractor in a time that increases as the system size grows; associated with the approach to the attractor is a diverging correlation length. Having reached the stationary state, further growth in the system takes place intermittently, in terms of bursts or avalanches which are correlated over all spatial and temporal extents. A prototypical example of SOC behavior in interfaces is the Sneppen model for depinning in a quenched random medium, where interface motion occurs at the weakest pinning center [4]. Since both Langevin dynamics and avalanche dynamics can describe critical behavior in systems out of equilibrium, it is natural to ask if there is any clear distinction between the two, or if they are, in fact, equivalent descriptions of the same phenomena.

*Electronic address: maya@cmt1.phy.bnl.gov

Here, I demonstrate that avalanche dynamics (SOC) can be distinguished phenomenologically from Langevin dynamics (generic scale invariance) by observing their dynamic scaling behaviors. A signature of SOC phenomena is a transient correlation length, in addition to the usual dynamic correlation length, that increases with time and diverges as the system reaches its stationary limit. Thus, unlike generically scale invariant systems, SOC systems exhibit different dynamic scaling behavior, with different critical exponents, during a transient self-organization process than in the stationary state. Both of these transient and stationary scaling regimes can, however, be unified into a single scaling function.

For simplicity, I will discuss this dynamic scaling approach in the context of interfacial growth and kinetic roughening phenomena, although the method is completely general. In order to make a clear distinction, I will first review the conventional description of kinetic roughening [9]. Consider a flat, d-dimensional surface of linear size L, with internal coordinate $\vec{x}$, at time $t=0$. This surface has a well-defined growth direction, and its configuration is represented by the single valued function $h(\vec{x},t)$ that only increases in time. The average height of the interface at time t is

$$\bar{h}(t)=\sum_{\text{all }\vec{x}}\frac{h(\vec{x},t)}{L^d}=\frac{s}{L^d}\,, \tag{1}$$

given an initial flat configuration, $h(\vec{x},t=0)=0$, $s(t=0)=0$ [10]. Here, the quantity s is the volume separating the current configuration from the initial flat configuration. The flat interface may roughen as it advances as a result of the buildup of random fluctuations. This roughening can be due to noise in the force which drives the growth, as in ballistic deposition, or can arise from defects and inhomogeneities in the medium through which the interface moves, as in interface depinning. Family and Vicsek [2] showed that a variety of stochastic interfacial growth models roughen according to a simple scaling form that can be written as follows:

$$W_0(L,t)=\left\langle\left[\frac{1}{L^d}\sum_{\text{all }\vec{x}}[h(x,t)-\bar{h}(t)]^2\right]^{1/2}\right\rangle\sim t^\beta F_0(t/L^z). \tag{2}$$

The mean-squared fluctuations in the height profile of the interface serve as a measure of its roughness. The brackets $\langle \cdots \rangle$ indicate an ensemble average over different realizations of randomness in the growth process. The crossover function

$$F_0(y) \sim \text{const} \qquad \text{for} \quad y \ll 1$$
$$F_0(y) \sim y^{-\beta} \qquad \text{for} \quad y \gg 1. \tag{3}$$

The coefficients β and z are the two critical exponents characterizing kinetic roughening processes. Eventually, for times $t \gg L^z$, the mean-squared height fluctuations reach a saturation width that has a power law dependence on system size L, $W_0 \sim L^{\chi}$. The roughness exponent obeys the scaling relation $\chi = z\beta$.

Similarly, one can consider how the mean-squared height fluctuations, $W(L,t_0,t+t_0)$, behave in the late stages of growth. Now, we compare height profiles between two configurations separated by time t, the first configuration at time t_0 and the second at a later time $t+t_0$. When $t_0=0$ this corresponds to W_0. In the limit $t_0 \to \infty$[11],

$$W(L,t_0,t_0+t) = \left\langle \left[\frac{1}{L^d} \sum_{\text{all } \vec{x}} \{h(x,t_0+t) - \bar{h}(t_0+t) - [h(x,t_0) - \bar{h}(t_0)]\}^2 \right]^{1/2} \right\rangle \sim t^{\beta} F_{\infty}(t/L^z) \ . \tag{4}$$

It is important to note that the function F_∞ has the same scaling limits as F_0

$$F_\infty(y) \sim \text{const} \qquad \text{for} \quad y \ll 1$$
$$F_\infty(y) \sim y^{-\beta} \qquad \text{for} \quad y \gg 1. \tag{5}$$

The upshot of this formalism is that kinetic roughening both in early stages and at late stages of growth is governed by the *same* two exponents β and z. There is one dynamic correlation length $\xi \sim t^{1/z}$ governing both the early and late stages of growth.

The reason is that many of these systems are described by the Kardar, Parisi, Zhang (KPZ) equation [6]:

$$\frac{\partial h}{\partial t} = \nabla^2 h + \frac{1}{2} \lambda (\nabla h)^2 + \eta(x,t), \tag{6}$$

or similar Langevin equations. In the KPZ equation, the variable η represents Gaussian distributed noise with zero mean. The coefficient λ accounts for lateral growth or sideways sticking in ballistic deposition. For $\lambda = 0$, one recovers the Edwards-Wilkinson equation for thermal roughening, which represents a different universality class. The KPZ equation is invariant with respect to translations in height, $h \to h + c$, or equivalently the overall amount of motion $s \to s + cL^d$, and time $t \to t + c$. This invariance requires that the universal critical exponents are independent of the amount of motion, or "history" of the interface. This is also the case when the noise has long range correlations in space and time [12], or is distributed according to a power law rather than being Gaussian [13].

One can choose to represent time in terms of the average height of the interface, $\bar{h} = s/L^d$. Substituting $t \to s/L^d$ and $h(x,t) \to h(x,s/L^d)$ gives a unified scaling form:

$$W(L,s_0,s_0+s) = \left\langle \left[\frac{1}{L^d} \sum_{\text{all } \vec{x}} [h(\vec{x},s_0+s) - h(\vec{x},s_0) - s/L^d]^2 \right]^{1/2} \right\rangle \sim (s/L^d)^{\beta} F(s_0/L^{d+z}, s/L^{z+d}), \tag{7}$$

with the boundary condition that $h(x,s_0=0)=0$. The scaling function $F(z,y)$ varies smoothly from early stages of growth, $z=0$, to late stages of growth, $z \to \infty$. The scaling behavior in both limits [Eqs. (3) and (5)] is the same. This formalism applies also to equilibrium dynamical phenomena because the initial condition breaks time translational invariance. Equation (7), with the smooth dependence of $F(z,y)$ on z, may possibly describe all generically scale invariant interfacial phenomena. Like equilibrium dynamical systems, they have the same universal scaling coefficients both in the early and late stages of growth.

Although the Family-Vicsek scaling form is widely applicable, it cannot describe all irreversible interfacial growth processes. Irreversible growth phenomena may have a dynamical attractor in much the same way that low-dimensional dynamical systems may have attractors. Associated with the approach to the attractor may be a second dynamic correlation length. This means that the behavior during the transient can be totally different than the behavior upon reaching the attractor.

For example, it has been demonstrated that a class of SOC systems, including the Sneppen model, are critical once they arrive in the stationary state, but have a finite correlation length for the dynamics during the transient [14–16]. For this class of models, it can be proven that the transient correlation length diverges as growth proceeds towards the stationary limit. This divergence of the transient correlation length will have a signature in the time dependent width; in particular the function $F(z,y)$ will have different scaling limits for small z than for $z \to \infty$. I propose that this is generally true for all SOC phenomena. The presence of a diverging correlation length for the transient can be observed using the dynamic scaling approach. This (second) diverging correlation length distinguishes SOC from generically scale invariant phenomena.

Actually, Sneppen and Jensen [17] noted that the Family-Vicsek scaling form did not describe SOC interface depinning in a simple model defined as follows [4]: An interface on a discrete lattice $(\vec{x},h)$ moves under the influence of quenched random pinning forces $\eta(\vec{x},h)$ assigned independently from a flat distribution in the unit interval. Growth occurs by advancing the extremal site on the interface with the smallest pinning force by one step, $h \to h+1$. Then neighboring sites are advanced to keep the nearest neighbor gradients less than 1, and the process is repeated. This dynamics represents a depinning transition at constant velocity rather than at constant driving force [16].

In order to proceed, it is convenient to represent dynamic scaling as follows:

$$W_q(L,s_0,s_0+s) = \left\langle \left(\frac{1}{L^d} \sum_{\text{all } \vec{x}} [h(\vec{x},s_0+s)-h(\vec{x},s_0)-s/L^d]^q \right)^{1/q} \right\rangle \sim (s/L^d)^{\beta} F_q(s_0/L^D, s/L^D). \quad (8)$$

The parameter q is included to allow for possible multifractal behavior of different moments. Multifractality in the stationary state of the Sneppen model [17] was explained by Tang and Leschhorn [18] and by Olami, Procaccia, and Zeitak [19]. It arises from the fact that for sufficiently small avalanches, most of the interface is frozen in a static configuration; only a small part of the interface actually moves forward.

The critical coefficient D appears as an "avalanche" dimension which controls the total amount of growth that must take place in order to reach the saturation regime. I first consider the behavior in the transient regime $s_0/L^D \ll 1$, where the function $F_q(z \ll 1, y) = f_{trans}(y)$. The scaling function

$$f_{trans}(y) \sim \text{const} \quad \text{for} \quad y \ll 1$$
$$f_{trans}(y) \sim y^{-\beta} \quad \text{for} \quad y \gg 1. \quad (9)$$

At early times, the transient width grows in time as $W \sim (s/L^d)^{\beta}$; while at late times the width saturates to an L dependent cutoff $W \sim L^{\beta(D-d)}$. This defines the roughness exponent for the saturation width

$$\chi = \beta(D-d). \quad (10)$$

In certain cases, the interface roughness exponent χ can be related to the avalanche dimension D. For interfaces with a compact avalanche dynamics, the amount of motion s during an avalanche scales with its projected area onto the substrate r^d multiplied by the linear extent $r_\perp \sim r^\chi$ in the direction of growth, so $s \sim r^d r_\perp \sim r^{d+\chi}$. This also requires that there is only one length scale for motion in the direction of growth. Every one of the r^d sites covered by the avalanche has finite probability to move $\sim r^\chi$ steps forward. This ansatz, which has been verified numerically for certain models [16,18–20], means that the avalanches have the same self-affine structure as the interface as a whole. For a system of size L, the avalanches will have a cutoff in their size $S_{co} \sim L^D \sim W L^d \sim L^{d+\chi}$, so

$$\chi = D-d \quad \text{and} \quad \beta = 1. \quad (11)$$

Actually, as shown later, $\beta = 1$ can be understood from a simple argument based on random deposition. Using different arguments, Olami, Procaccia, and Zeitak [19] derived $\beta = 1$ for the special case of the one-dimensional Sneppen model, where $\beta \approx 1$ has been measured [4]. Scaling theory, Eq. (10), suggests that $\beta = 1$ holds for any SOC interface model in any dimension as long as the avalanche dimension $D = d + \chi$.

In the stationary regime, $s_0/L^d \gg 1$, the scaling function has a *different* limit than in the transient regime and also exhibits multifractality; $F_q(z \gg 1, y) = y^{\alpha(q)} f_q(y)$. The scaling function

$$f_q(y) \to \text{const} \quad \text{for} \quad y \ll 1$$
$$f_q(y) \to y^{-\alpha(q)-\beta} \quad \text{for} \quad y \gg 1. \quad (12)$$

For small s the width grows as $W \sim (s/L^d)^{\beta} (s/L^D)^{\alpha(q)}$; while at long times s, it saturates to the same time independent width as in the transient regime, $W \sim L^{\beta(D-d)}$. If the activity in the stationary regime takes place in terms of avalanches, which are bursts over localized regions, the sum over all lattice sites in Eq. (8) can be replaced with a sum over the localized area spanned by the avalanche. Since most of the sites in the system are frozen during the avalanche, this sum does not depend on L. As a result, the L dependence of the dynamical width must be $\sim L^{-d/q}$, for $s \ll L^D$. Thus

$$\alpha(q) = \frac{d}{D}\left(\frac{1}{q} - \beta\right), \quad (13)$$

and the width in the stationary state grows as

$$W_q \sim s^{\beta((D-d)/D)+(d/qD)} L^{-d/q}. \quad (14)$$

The dynamic exponent for time dependent roughening in the stationary state,

$$\beta_{sta}(q) = \frac{\beta(D-d)}{D} + \frac{d}{qD}, \quad (15)$$

is different than the dynamic exponent β for time dependent roughening during the transient. If $\chi = D-d$ and correspondingly $\beta = 1$ then

$$W_q \sim s^{[(d+q\chi)/q(d+\chi)]} L^{-d/q} \quad \text{or} \quad \beta_{sta}(q) = \frac{d+q\chi}{q(d+\chi)}. \quad (16)$$

Based on similar arguments, the above results should also hold for an SOC interface model where the force at each site on the interface is given by a quenched version of the KPZ equation:

$$F(\vec{x},h) = \nabla^2 h + \lambda(\nabla h)^2 + \eta(\vec{x},h), \quad (17)$$

and the site with the largest force F is advanced at each sequential time step, $s \to s+1$. Here $\eta(\vec{x},h)$ represents quenched random pinning forces. For finite λ, this model is believed to be in the same universality class as the Sneppen model. For $\lambda = 0$, it is in a different universality class which may describe fluid invasion in a porous medium [15,19,16].

Recently Schmittbuhl *et al.* introduced a nonlocal model for SOC contact line depinning and crack propagation in a quenched random medium [21]. In this case, the local diffusive and nonlinear terms in Eq. (17) are replaced by a nonlocal kernel which decays as $1/x^2$. Despite the nonlocality, the model appears to have the same phenomenology of SOC and avalanches as the Sneppen model. These authors describe the scaling behavior using the Family-Vicsek scaling form which, as emphasized here, does not distinguish between transient and stationary state behavior in SOC systems. It is important to note that they found an exponent $\beta \approx 1/2$ rather than $\beta = 1$. This exponent was obtained using data-collapse techniques for small systems $L = 32$–1024, rather than by direct measurement of temporal scaling of the width. It would be useful also to compare a direct measure-

ment of the avalanche dimension D with their independent measurement of the surface roughness χ. If $D=d+\chi$, then $\beta=1$ in their model. It would also be interesting to test these results on other models.

The result $\beta=1$ derived from scaling theory reflects a self-organization process that has a simple and plausible explanation. Consider an SOC interface model, such as Eq. (17) starting from a flat configuration. Initially, growth occurs randomly throughout the system, and for early times, the mean-squared-width grows with the number of sequential advances according to a Poisson distribution, as in the case of random deposition. After some time, s, correlations have built up in the system so that the bursts will have a characteristic size $s_{co}(s)$, spatial extent $r_{co}(s)$, and width $w_{co}(s)$. At length or time scales larger than these cutoff scales, the process will still be Poissonian because because the system has not yet organized itself at these scales. Initially growth occurs by adding blocks of linear extent a and vertical extent a, corresponding to the lattice unit size. For random deposition the mean-squared-width increases with the number of blocks "dropped" onto the system as

$$W^2=\frac{(\text{number of drops})}{(\text{site})}(\text{vertical size of block})^2. \tag{18}$$

Initially, then

$$W^2=\left(\frac{s}{L^d}\right)a^2. \tag{19}$$

This relation will be valid as long as the number of drops per site is small. As the interface advances, this number obviously increases, so the system is no longer Poissonian at the scale of the lattice unit. However, the process can still be described as a random deposition process under a suitable coarse-graining procedure where the sites and blocks dropping upon them are increased in linear size to be $r_{co}(s)$. In the rescaled units, each site will still have a small number of blocks. Each block has a mass $s_{co}\sim r_{co}^D$. Equations (18) and (19) now hold if s is rescaled by s_{co}, L is rescaled by r_{co}, and the block vertical extent is increased to $w_{co}\sim r_{co}^{\chi}$. This leads to

$$W^2=\left(\frac{\left(\frac{s}{s_{co}}\right)}{\left(\frac{L}{r_{co}}\right)^d}\right)(w_{co}a)^2. \tag{20}$$

The dependence $r_{co}(s)$ is determined by the requirement that the number of drops per site is of order unity in the coarse graining procedure; in order to maintain the Poisson distribution,

$$\frac{\left(\frac{s}{s_{co}}\right)}{\left(\frac{L}{r_{co}}\right)^d}\sim 1 \quad \text{or} \quad r_{co}\sim\left(\frac{s}{L^d}\right)^{1/(D-d)}. \tag{21}$$

Note that r_{co} is actually the transient correlation length that leads to different scaling behaviors in the early and late stages of growth. Using this result, I find

$$W^2\sim\left(\frac{s}{L^d}\right)^{2\beta}\sim\left(\frac{s}{L^d}\right)^{2\chi/(D-d)}. \tag{22}$$

As long as the avalanches have a compact geometrical structure that is characterized solely by the spatial extent r_{co} and the extension in the direction of motion, $w_{co}\sim r_{co}^{\chi}$, then $\chi=D-d$ and $\beta=1$ for a class of SOC interfacial phenomena.

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